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A NEW TOOL FOR THE LAMB-SHIFT CALCULATION

Marco Cavicchi[1],
Antonio Vairo[2]

I.N.F.N. - *Sez. di Bologna and Dip. di Fisica,*
Università di Bologna, Via Irnerio 46, I-40126 Bologna, Italy

Abstract

We solve the Bethe-Salpeter equation for hydrogenic bound states by choosing an appropriate interaction kernel K_c. We want to use our solution to calculate up to a higher order the hydrogen Lamb-shift, and as a first application we present up to order $(\alpha/\pi)(Z\alpha)^7$ the contribution of the lowest order self-energy graph, calculated *exactly*. The basic formalism is a natural extension to the hydrogenic bound states of the one previously presented by R. Barbieri and E. Remiddi and used in the case of positronium.

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[1] E-mail: cavicchi@bologna.infn.it / cavicchi@nbivax.nbi.dk
[2] E-mail: vairo@bologna.infn.it

1   Introduction

Bound state systems like positronium and muonium are a good test of QED. Hydrogenic atoms are not completely reducible to a QED problem because the structure of the proton. In particular the finite size of the proton gives rise to a lower limit of the theoretical calculus precision (for a good review see [1] and more recently [2]).

However, there are two kind of considerations which make hydrogenic atoms still interesting from a QED point of view. First, the requested precision is not yet achieved in many theoretical predictions, e.g. the hydrogen Lamb-shift requires a theoretical precision of about 1 kHz (the present status of the calculation of the hydrogen Lamb-shift can be found in [3]), and new but incomplete contributions have been recently calculated (e.g. [4], [5]). Second, the fully detailed treatment of a relatively simpler problem as the hydrogenic atom (which is reducible in QED to a one particle problem in an external potential) can give a hint for the treatment of more complicated system as the positronium. For this purpose it is necessary to use the same formalism to describe the positronium and the hydrogenic atoms.

In sect.2 we extend to the hydrogenic atoms the formalism which was proposed by R. Barbieri and E. Remiddi [6] for the positronium (which we call BR formalism). Following [6] we propose as interaction kernel a sort of *relativistic dressed* Coulomb interaction, so that the Bethe-Salpeter equation, which for bare Coulomb interaction is the Dirac-Coulomb equation, is analytically solvable in closed form. Then we write a perturbative expansion for the energy levels which immediately reproduces the Dirac levels. In sect.3, as a first application of our formalism, we calculate analytically up to order $\alpha/\pi(Z\alpha)^7$ for the levels $n = 1, 2$ the first contribution to the self-energy together with other graphs necessary to cancel spurious terms, which typically arise in this sort of calculation. An appendix is devoted to review and discuss the method proposed by J. Sapirstein [7] to treat perturbatively the Dirac propagator.

2 BR formalism in the context of the hydrogenic atoms

In QED hydrogenic atoms are well approximated by an electron moving in an external field $V_c(r) = -Z\alpha/r$, that represents his Coulomb interaction with the nucleus. The Green's function $G(W; \vec{p}, \vec{q})$ of this electron contains all the necessary informations about the system, and it can be written in perturbation theory as the sum of all the Feynman's graphs with an incoming and an outcoming electron leg (see fig.1; W is the energy and $\vec{p}, \vec{q}$ are the incoming and outcoming spatial components of the electron's momentum; the ball includes all radiative corrections, the external lines are Coulomb interaction vertices).
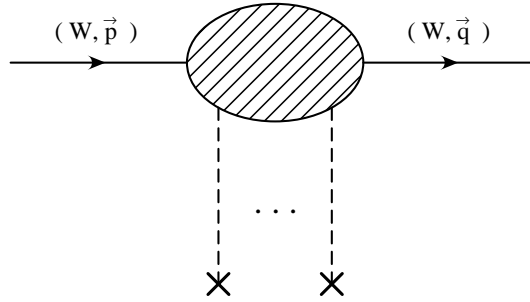

Fig.1 *The Green's function $G(W; \vec{p}, \vec{q})$ of an electron in external field.*

It is well known that in bound state theory the Green function G has simple poles in the energy W. For hydrogenic atoms these poles are grouped round the Balmer levels $-m(Z\alpha)^2/2n^2$ and are labeled by the quantum numbers n, l (the angular momentum which is an " almost good quantum number" in the sense of [8]) and j (the total momentum). For every pole W_{nlj} we can write:

$$G(W; \vec{p}, \vec{q}) = \frac{R_{nlj}(\vec{p}, \vec{q})}{W - W_{nlj}} + \hat{G}_{nlj}(W; \vec{p}, \vec{q}), \qquad (2.1)$$

where R_{nlj} is the residuum and $\hat{G}_{nlj}$ is the regular part of G at $W = W_{nlj}$.

The standard way to obtain a perturbative expansion of W_{nlj} starts from the Bethe-Salpeter equation [9]:

$$
\begin{aligned}
G(W; \vec{p}, \vec{q}) \;=\; & G_0(W; \vec{p}) \Big[(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\
& + \int \frac{d^3 k}{(2\pi)^3} K(W; \vec{p}, \vec{k}) G(W; \vec{k}, \vec{q}) \Big],
\end{aligned}
\tag{2.2}
$$

where $G_0(W; \vec{p}) = i(\gamma^0 W - \vec{p} \cdot \vec{\gamma} - m + i\epsilon)^{-1}$ is the electron free propagator and $K(W; \vec{p}, \vec{q})$ is the interaction kernel (i.e. the sum of all 1-particle irreducible graphs with external fermionic legs removed). Then we write:

$$
K(W; \vec{p}, \vec{q}) \equiv K_c(W; \vec{p}, \vec{q}) + \delta K(W; \vec{p}, \vec{q}),
\tag{2.3}
$$

which is, rigorously speaking, a definition of δK, once K_c is suitably chosen. To choose K_c, we ask that it satisfies the two conditions:
(i) in the non relativistic limit $K_c \to -i\gamma_0 V_c$;
(ii) the Bethe-Salpeter equation for K_c:

$$
\begin{aligned}
G_c(W; \vec{p}, \vec{q}) \;=\; & G_0(W; \vec{p}) \Big[(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\
& + \int \frac{d^3 k}{(2\pi)^3} K_c(W; \vec{p}, \vec{k}) G_c(W; \vec{k}, \vec{q}) \Big],
\end{aligned}
\tag{2.4}
$$

can be explicitly solved in closed form.

From condition (i) it follows that G_c can be considered as a zeroth order approximation of G. Therefore G_c has singularities in $W = W_n^c$ [1] and $W_n^c \approx m - m(Z\alpha)^2/2n^2$:

$$
G_c(W; \vec{p}, \vec{q}) = \frac{\sum_{lj} R_{nlj}^c(\vec{p}, \vec{q})}{W - W_n^c} + \hat{G}_n^c(W; \vec{p}, \vec{q}),
\tag{2.5}
$$

where $\sum_{lj} R_{nlj}^c$ is the residuum and $\hat{G}_n^c$ is the regular part of G_c at $W = W_n^c$. From condition (ii) it follows that W_n^c, R_{nlj}^c and $\hat{G}_n^c$ are explicitly known.

---

[1]According to general expectations the exact problem has less degeneracy than the unperturbed one.

Using the quantities defined above, first we can write the Bethe-Salpeter equation (2.2) in terms of G_c and δK as:

$$G(W;\vec{p},\vec{q}) = G_c(W;\vec{p})\Big[(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$$
$$+ \int \frac{d^3k}{(2\pi)^3}\delta K(W;\vec{p},\vec{k})G(W;\vec{k},\vec{q})\Big], \qquad (2.6)$$

and then the perturbative expansion of the energy levels:

$$W_{nlj} = W_n^c + \frac{1}{D}Tr\left\{\delta K(W_n^c)R_{nlj}^c\right\}$$
$$+ \frac{1}{D^2}Tr\left\{\delta K(W_n^c)\hat{G}_n^c(W_n^c)\delta K(W_n^c)R_{nlj}^c\right\} \qquad (2.7)$$
$$+ \frac{1}{D^2}Tr\left\{\delta K(W_n^c)R_{nlj}^c\right\}Tr\left\{\frac{\partial}{\partial W}\delta K(W_n^c)R_{nlj}^c\right\} + \dots ,$$

where we have omitted for simplicity the explicit indication of the momenta, and D is defined as the following trace [2]:

$$D \equiv Tr\left\{\frac{\partial}{\partial W}G_c^{-1}(W_n^c)R_{nlj}^c\right\}. \qquad (2.8)$$

Each term of (2.7) is a series in $(Z\alpha)$ with fixed (α/π). We observe that the expansion (2.7) is "perturbative" in the sense of increasing orders of $\delta K(W_n^c)$. For consistency the explicit calculation must exhibit that to an increasing order in $\delta K(W_n^c)$ it corresponds an increasing leading order in $Z\alpha$. In [10] the reader can find a more detailed discussion.

In [11], [6] (see also [12]) it was found a kernel K_c satisfying conditions (i)

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[2] From (2.4) it follows that $G_c^{-1} = G_0^{-1} - K_c$ and then we can write (2.8) also as:

$$D = -Tr\left\{i\gamma^0 R_{nlj}^c\right\} - Tr\left\{\frac{\partial}{\partial W}K_c(W_n^c)R_{nlj}^c\right\}.$$

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and (ii) for positronium. In what follows we choose for K_c [3]:

$$K_c(W; \vec{p}, \vec{q}) = -i \ f(W; p, q)\tilde{V}_c(\vec{p} - \vec{q})\gamma^0\Lambda_+(\vec{p})\frac{1 + \gamma^0}{2}\Lambda_+(\vec{q}), \qquad (2.9)$$

where $\tilde{V}_c(\vec{p}) = -4\pi Z\alpha/\vec{p}^2$ is the Fourier transform of the Coulomb potential, and

$$\Lambda_\pm(\vec{p}) = \frac{E_p \pm (m - \vec{p} \cdot \vec{\gamma})\gamma^0}{2E_p},$$

$$f(W; p, q) = \left(\frac{16m^2 E_p E_q}{(E_p + m)(E_p + W)(E_q + m)(E_q + W)}\right)^{\frac{1}{2}},$$

$$E_p = \sqrt{\vec{p}^{\,2} + m^2}.$$

In the non relativistic limit $\vec{p}, \vec{q} \longrightarrow 0$ it is easily seen that K_c satisfies condition (i).

If we define $H_c(W; \vec{p}, \vec{q})$ as:

$$G_c(W; \vec{p}, \vec{q}) \equiv G_0(W; \vec{p})\Bigg[(2\pi)^3\delta^{(3)}(\vec{p} - \vec{q})$$

$$-i \ f(W; p, q)\gamma^0\Lambda_+(\vec{p})\frac{1 + \gamma^0}{2}\Lambda_+(\vec{q})G_0(W; \vec{q})H_c(W; \vec{p}, \vec{q})\Bigg];$$

$$(2.10)$$

from the comparison of (2.10) and (2.4) we obtain the following equation for H_c:

$$H_c(W; \vec{p}, \vec{q}) = \tilde{V}_c(\vec{p} - \vec{q}) \qquad\qquad\qquad (2.11)$$

$$+ \int \frac{d^3k}{(2\pi)^3}\frac{1}{\frac{W^2 - m^2}{2m} - \frac{\vec{k}^{\,2}}{2m} + i\epsilon}\tilde{V}_c(\vec{p} - \vec{k})H_c(W; \vec{k}, \vec{q});$$

---

[3] An other choice can be $K_c = K_D \equiv -i\gamma^0\tilde{V}_c$. As a consequence of this choice $G_c = g_D$ (where g_D is the Dirac–Coulomb propagator and will be defined later). g_D is known in analytical closed form and its poles and residuum at the poles also (see [13] and for a detailed study [14]). But analytical computation with these quantities are more complicate than in our formalism because the unpractical structure of g_D.

the solution of (2.11) is known and can be written in the Schwinger integral representation [15]:

$$
\begin{aligned}
H_c(W; \vec{p}, \vec{q}) &= \tilde{V}_c(\vec{p} - \vec{q}) + (Z\alpha)^2 \frac{1}{(\vec{p} - \vec{q})^2} \frac{4\pi m}{\sqrt{m^2 - W^2}} \\
&\cdot \int_0^1 d\rho \frac{\rho^{\frac{-mZ\alpha}{\sqrt{m^2 - W^2}}}}{\rho + \frac{(E_p^2 - W^2)(Eq^2 - W^2)}{4(m^2 - W^2)(\vec{p} - \vec{q})^2}(1 - \rho)^2}.
\end{aligned}
\tag{2.12}
$$

Inserting (2.12) into (2.10) we obtain:

$$
\begin{aligned}
G_c(W; \vec{p}, \vec{q}) &= (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) G_0(W; \vec{p}) \\
&+ i \frac{f(W; p, q)}{(E_p - W)(E_q - W)} \Lambda_+(\vec{p}) \frac{1 + \gamma^0}{2} \Lambda_+(\vec{q}) \gamma^0 \tilde{V}_c(\vec{p} - \vec{q}) \\
&\cdot \left[1 + Z\alpha \frac{m}{\sqrt{m^2 - W^2}} \int_0^1 d\rho \frac{\rho^{\frac{-mZ\alpha}{\sqrt{m^2 - W^2}}}}{\rho + \frac{(E_p^2 - W^2)(E_q^2 - W^2)}{4(m^2 - W^2)(\vec{p} - \vec{q})^2}(1 - \rho)^2} \right],
\end{aligned}
\tag{2.13}
$$

so that condition (ii) is also fulfilled.

From (2.13), we see how the Green function G_c has poles at the values of W

$$
W_n^c = m\sqrt{1 - \frac{(Z\alpha)^2}{n^2}},
\tag{2.14}
$$

and isolating the singular part from the regular one at $W = W_n^c$ in (2.13) we find also the residuum $\sum_{lj} R_{nlj}^c$ at the pole:

$$
\begin{aligned}
\sum_{lj} R_{nlj}^c(\vec{p}, \vec{q}) &= \frac{i}{W_n^c} \left(\frac{E_p(E_p + W_n^c)}{E_p + m} \right)^{\frac{1}{2}} \left(\frac{E_q(E_q + W_n^c)}{E_q + m} \right)^{\frac{1}{2}} \\
&\cdot \Lambda_+(\vec{p}) \frac{1 + \gamma^0}{2} \sum_{l=0}^{n-1} R_{nl}(p) R_{nl}(q) \frac{2l + 1}{4\pi} P_l\left(\frac{\vec{p} \cdot \vec{q}}{pq} \right) \Lambda_+(\vec{q}) \gamma^0,
\end{aligned}
\tag{2.15}
$$

where $P_l(z)$ is the Legendre polynomial of order l and R_{nl} the radial part of the Schrödinger-Coulomb wave functions.

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$\hat{G}_n^c(W; \vec{p}, \vec{q})$, the regular part of G_c at $W = W_n^c$, can be obtained explicitly subtracting from (2.13) the singular part of (2.5) and taking (2.15) into account (see [16] for the explicit case of positronium).

To obtain from (2.15) the expression of R_{nlj}^c, following [10], we write the identity:

$$\frac{2l+1}{4\pi} P_l\left(\frac{\vec{p}\cdot\vec{q}}{pq}\right) = \sum_{j=|l-\frac{1}{2}|}^{j=l+\frac{1}{2}} D_l^j\left(\frac{\vec{p}}{p}, \frac{\vec{q}}{q}\right), \qquad (2.16)$$

where

$$\begin{aligned} D_l^j\left(\frac{\vec{p}}{p}, \frac{\vec{q}}{q}\right) &= \frac{1}{4\pi}\left(j+\frac{1}{2}\right) P_l\left(\frac{\vec{p}\cdot\vec{q}}{pq}\right) \\ &+ \frac{j-l}{2\pi}\left(\frac{\vec{p}\cdot\vec{\gamma}}{p}\frac{\vec{q}\cdot\vec{\gamma}}{q} + \frac{\vec{p}\cdot\vec{q}}{pq}\right)\frac{\partial}{\partial z}P_l(z)\bigg|_{\cos z=\frac{\vec{p}\cdot\vec{q}}{pq}}. \end{aligned}$$

R_{nlj}^c is then given by:

$$\begin{aligned} R_{nlj}^c(\vec{p}, \vec{q}) &= \frac{i}{W_n^c}\left(\frac{E_p(E_p + W_n^c)}{E_p + m}\right)^{\frac{1}{2}}\left(\frac{E_q(E_q + W_n^c)}{E_q + m}\right)^{\frac{1}{2}} \qquad (2.17) \\ &\cdot \ R_{nl}(p)R_{nl}(q)\Lambda_+(\vec{p})\frac{1+\gamma^0}{2}D_l^j\left(\frac{\vec{p}}{p}, \frac{\vec{q}}{q}\right)\Lambda_+(\vec{q})\gamma^0, \end{aligned}$$

which is an eigenfunction of parity and total momentum. R_{nlj}^c satisfies:

$$R_{nlj}^c(\vec{p}, \vec{q}) = G_0(W_n^c; \vec{p})\int \frac{d^3k}{(2\pi)^3}K_c(W_n^c; \vec{p}, \vec{k})R_{nlj}^c(\vec{k}, \vec{q}), \qquad (2.18)$$

known as the Bethe-Salpeter equation for the residuum.

The expansion (2.7) is now completely defined. The first terms of the expansion that must be calculated are D and $Tr\left\{K_c(W_n^c)R_{nlj}^c\right\}$. These terms are not related to any Feynman graph, they only depend on the formalism which we have adopted.

After explicit calculation we find:

$$D = 2j + 1, \qquad (2.19)$$

and

$$\langle K_c \rangle_n \equiv \frac{1}{D} Tr \left\{ K_c(W_n^c) R_{nlj}^c \right\} = -\frac{m^2}{W_n^c} \frac{(Z\alpha)^2}{n^2}. \qquad (2.20)$$

D is the degeneration of the level (n, l, j) [4] and $\langle K_c \rangle_n$ is, at the leading order (apart for a factor 2), the Balmer series (which follows from condition (i) on K_c).

The next term of the expansion (2.7) giving the leading corrections to the level is $Tr \left\{ K(W_n^c) R_{nlj}^c \right\}$. In fig.2 we show K up to one loop radiative corrections.
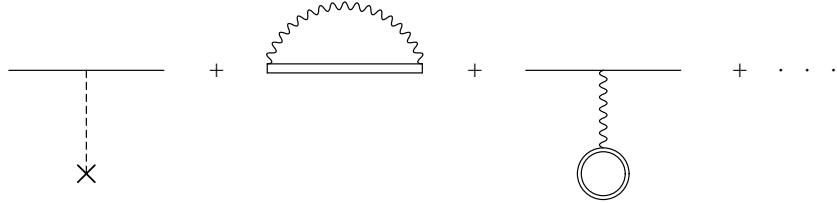

Fig.2 *The first terms of the irreducible kernel K for hydrogenic atoms.*

In fig.2 with the doubled line we have represented the sum of graphs of fig.3. These graphs can be resummed using the Dirac-Coulomb equation:

$$\begin{aligned} g_D(W; \vec{p}, \vec{q}) &= G_0(W; \vec{p}) \bigg[(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \\ &+ \int \frac{d^3 k}{(2\pi)^3} \left(-i\tilde{V}_c(\vec{p} - \vec{k})\gamma^0 \right) g_D(W; \vec{k}, \vec{q}) \bigg]. \qquad (2.21) \end{aligned}$$

This equation is of the kind of the Bethe-Salpeter equation (2.2) with kernel:

$$K_D(\vec{p}, \vec{q}) = -i\tilde{V}_c(\vec{p} - \vec{q})\gamma^0. \qquad (2.22)$$

A graphical representation of K_D is given by the first graph of fig.2. Therefore equation (2.21) can be treated perturbatively in the formalism of equations

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[4]This result less than surprising is a natural consequence of the Bethe-Salpeter formalism.

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(2.2)-(2.6) with K_c given by equation (2.9) and consequently $\delta K = K_D - K_c$. An alternative approach to treat perturbatively the Dirac-Coulomb propagator g_D was given by J. Sapirstein [7] and can be found in appendix.
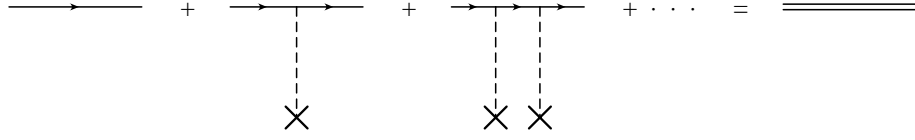

Fig.3 *The Dirac-Coulomb propagator g_D.*

The contribution to the energy levels of K_D originates up to order $(Z\alpha)^4$ the well-known Dirac levels W_{nj}^D:

$$\langle K_D \rangle_{nlj} \equiv \frac{1}{D} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} Tr\left\{-i\tilde{V}_c(\vec{p}-\vec{q})\gamma^0 R_{nlj}^c(\vec{p},\vec{q})\right\}$$

$$= -m\frac{(Z\alpha)^2}{n^2} - m\frac{(Z\alpha)^4}{2n^3}\left(\frac{1}{j+\frac{1}{2}}\right), \tag{2.23}$$

$$W_{nj}^D = W_n^c + \langle K_D \rangle_{nlj} - \langle K_c \rangle_n$$

$$= m - m\frac{(Z\alpha)^2}{2n^2} - m\frac{(Z\alpha)^4}{2n^3}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right). \tag{2.24}$$

More interesting are the one loop corrections. In particular in the next section we discuss the contribution to (2.7) of the second graph of fig.2, the self-energy graph.

3 Application to the self-energy contribution

The first one-loop correction to the hydrogenic atoms' energy levels is given by the self-energy graph (second graph of fig.2; a recent calculation of this

contribution in an other method can be found in [17]).

Let us recall that the contribution to the self-energy kernel due to the first graph of the right hand side of fig.4 could be written as:

$$i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4}(-ie\gamma^\mu)\frac{i}{\not p - \not k - m + i\epsilon}(-ie\gamma^\nu)\frac{-ig_{\mu\nu}}{k^2 + i\epsilon}. \qquad (3.1)$$
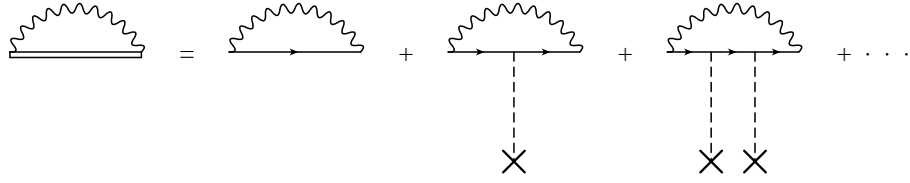

Fig.4 *The $(Z\alpha)$ expansion for the self-energy graph.*

Expression (3.1) is obviously divergent. Choosing the Pauli-Villars regularization one has:

$$
\begin{aligned}
i\,\Sigma_{reg}(p;\Lambda) &= \int \frac{d^4k}{(2\pi)^4}(-ie\gamma^\mu)\frac{i}{\not p - \not k - m + i\epsilon}(-ie\gamma^\nu)\\
&\quad \cdot \frac{-ig_{\mu\nu}}{k^2 + i\epsilon}\frac{-\Lambda^2}{k^2 - \Lambda^2 + i\epsilon}\\
&= i\,\frac{\alpha}{\pi}[(-\not p + m)A(p^2;\Lambda) + m\,B(p^2;\Lambda)], \qquad (3.2)
\end{aligned}
$$

where

$$
\begin{aligned}
A(p^2;\Lambda) &= \frac{1}{4}(1 - m^2/p^2)(1 + m^2/p^2)\log(1 - p^2/m^2) - \frac{1}{4}\left(\frac{m^2}{p^2} - 1\right)\\
&\quad - \frac{5}{8} - \frac{1}{4}\log\left(\frac{\Lambda^2}{m^2}\right),\\
B(p^2;\Lambda) &= \frac{1}{4}(1 - m^2/p^2)(3 - m^2/p^2)\log(1 - p^2/m^2) + \frac{1}{4}\left(\frac{m^2}{p^2} - 1\right)\\
&\quad - \frac{3}{8} - \frac{3}{4}\log\left(\frac{\Lambda^2}{m^2}\right). \qquad (3.3)
\end{aligned}
$$

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The mass-shell renormalization prescription is imposed by defining the mass renormalization counterterm:

$$i \; \delta m \equiv -i \Sigma_{reg}(p; \Lambda) \Big|_{\not{p}=m} \tag{3.4}$$

so that [13]

$$i \; \delta m = i \frac{\alpha}{\pi} m \left[\frac{3}{8} + \frac{3}{4} \log \left(\frac{\Lambda^2}{m^2} \right) \right] \tag{3.5}$$

The relevant kernel for our purpose is the regularized and mass-subtracted graph:

$$K_{s.e.}(p; \Lambda) = i \left(\Sigma_{reg}(p; \Lambda) + \delta m \right), \tag{3.6}$$

which is still u.v. divergent (wave-function renormalization has not been carried out). If we rearrange the terms to isolate the Λ dependent part $K_{s.e.}^{div}$ from the rest $\hat{K}_{s.e.}$, we will obtain:

$$K_{s.e.}(p; \Lambda) = K_{s.e.}^{div}(p; \Lambda) + \hat{K}_{s.e.}(p) \tag{3.7}$$

where

$$
\begin{aligned}
K_{s.e.}^{div}(p; \Lambda) &= i \frac{\alpha}{\pi} (-\not{p} + m) \left(-\frac{1}{4} \log \left(\frac{\Lambda^2}{m^2} \right) \right) \\
&= \frac{\alpha}{\pi} G_0^{-1}(p) \left(-\frac{1}{4} \log \left(\frac{\Lambda^2}{m^2} \right) \right),
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
\hat{K}_{s.e.}(p; \Lambda) &= i \frac{\alpha}{\pi} [(-\not{p} + m) \hat{A}(p^2) + m \; \hat{B}(p^2)], \\
\hat{A}(p^2) &= \frac{1}{4} \left(1 - \frac{m^2}{p^2} \right) \left(1 + \frac{m^2}{p^2} \right) \log(1 - p^2/m^2) \\
&\quad - \frac{1}{4} \left(\frac{m^2}{p^2} - 1 \right) - \frac{5}{8}, \\
\hat{B}(p^2) &= \frac{1}{4} \left(1 - \frac{m^2}{p^2} \right) \left(3 - \frac{m^2}{p^2} \right) \log(1 - p^2/m^2) \\
&\quad + \frac{1}{4} \left(\frac{m^2}{p^2} - 1 \right).
\end{aligned} \tag{3.9}
$$

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The contribution of (3.7) to the energy levels according to (2.7) is given by:

$$
\begin{aligned}
\langle K_{s.e.}\rangle_{nlj} &= \left\langle K_{s.e.}^{div}\right\rangle_{nlj} + \left\langle \hat{K}_{s.e.}\right\rangle_{nlj} \\
&\equiv \frac{1}{D}Tr\left\{K_{s.e.}^{div}(W_n^c)R_{nlj}^c\right\} + \frac{1}{D}Tr\left\{\hat{K}_{s.e.}(W_n^c)R_{nlj}^c\right\}.
\end{aligned}
\tag{3.10}
$$

Because R_{nlj}^c satisfies the Bethe-Salpeter equation (2.18) and taking in account equation (3.8) we can easily calculate $\left\langle K_{s.e.}^{div}\right\rangle_{nlj}$:

$$
\begin{aligned}
\left\langle K_{s.e.}^{div}\right\rangle_{nlj} &= \frac{1}{D}Tr\left\{K_{s.e.}^{div}R_{nlj}^c\right\} = \frac{1}{D}Tr\left\{K_{s.e.}^{div}G_0K_cR_{nlj}^c\right\} \\
&= \frac{\alpha}{\pi}\left(-\frac{1}{4}\log\left(\frac{\Lambda^2}{m^2}\right)\right)\frac{1}{D}Tr\left\{K_cR_{nlj}^c\right\} \\
&= \frac{\alpha}{\pi}\left(-\frac{1}{4}\log\left(\frac{\Lambda^2}{m^2}\right)\right)\langle K_c\rangle_n.
\end{aligned}
\tag{3.11}
$$

The explicit value of $\langle K_c\rangle_n$ has been given in (2.20).

We don't need to perform a wave-function renormalization: if we adopt a gauge invariant regularization scheme the wave-function renormalization counterterm is exactly compensated by the vertex renormalization counterterm, hence one obtains the correct physical result without taking in account these counterterms [5]. The Λ dependent term (3.11) must also cancel if we evaluate the vertex contribution to the energy expansion (2.7) in the same gauge invariant Pauli-Villars regularization scheme adopted for the self-energy and without subtracting the renormalization counterterm (see for a detailed discussion in the positronium context [12]).

We have therefore to consider the second graph of the right hand side of fig.4. The kernel due to this graph is:

$$
K_{ver} = -i\tilde{V}_c\Gamma_{reg}^0,
\tag{3.12}
$$

where the Pauli-Villars regularized Γ_{reg}^μ,

$$
\Gamma_{reg}^\mu(p,q;\Lambda) = \int\frac{d^4k}{(2\pi)^4}(-ie\gamma^\rho)\frac{i}{\not{p}-\not{k}-m+i\epsilon}\gamma^\mu\frac{i}{\not{q}-\not{k}-m+i\epsilon}(-ie\gamma^\sigma)
$$

---

[5]Moreover in this manner one avoids the problem of the spurious infrared divergences which may arise from mass-shell wave-function renormalization.

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$$\cdot \quad \frac{-ig_{\rho\sigma}}{k^2 + i\epsilon} \frac{-\Lambda^2}{k^2 - \Lambda^2 + i\epsilon}, \tag{3.13}$$

satisfies the Ward identity:

$$(p-q)_\mu \Gamma^\mu_{reg}(p, q; \Lambda) = \Sigma_{reg}(p; \Lambda) - \Sigma_{reg}(q; \Lambda). \tag{3.14}$$

From (3.14),(3.2) and (3.3), differentiating with respect to p_ν and rearranging terms, it follows that:

$$\begin{aligned}
\Gamma^\nu_{reg}(p, q; \Lambda) &= \frac{\partial}{\partial p_\nu} \Sigma_{reg}(p; \Lambda) - (p-q)_\mu \frac{\partial}{\partial p_\nu} \Gamma^\mu_{reg}(p, q; \Lambda) \\
&= -\frac{\alpha}{\pi} \gamma^\nu \left(-\frac{1}{4} \log \left(\frac{\Lambda^2}{m^2} \right) \right) + \hat{\Gamma}^\nu(p, q),
\end{aligned} \tag{3.15}$$

where $\hat{\Gamma}^\nu$ is u.v. finite (Λ independent).
The contribution of (3.12) to the energy levels is also given by:

$$\begin{aligned}
\langle K_{ver} \rangle_{nlj} &\equiv \frac{1}{D} Tr \left\{ -i\tilde{V}_c \Gamma^0_{reg}(W^c_n) R^c_{nlj} \right\} \tag{3.16} \\
&= -\frac{\alpha}{\pi} \left(-\frac{1}{4} \log \left(\frac{\Lambda^2}{m^2} \right) \right) \langle K_D \rangle_{nlj} + \left\langle \hat{K}_{ver} \right\rangle_{nlj},
\end{aligned}$$

where we have defined $\hat{K}_{ver} \equiv -i\tilde{V}_c \hat{\Gamma}^0$. The explicit value of $\langle K_D \rangle_{nlj}$ has been given in (2.23).

Comparing (3.11) with (3.16) we note that only the $\alpha/\pi(Z\alpha)^2 \log(\Lambda^2/m^2)$ terms, corresponding to the leading order in $Z\alpha$, cancel. Then for a complete cancellation of the divergent terms it is necessary to sum other divergent terms arising from expansion (2.7) [6]. To obtain for instance the cancellation up to order $\alpha/\pi(Z\alpha)^4 \log(\Lambda^2/m^2)$ the terms which must be considered arise from the last contribution explicitly written in expansion (2.7):

$$\langle \delta K \rangle_{nlj} \left\langle \frac{\partial}{\partial W} \delta K \right\rangle_{nlj} \equiv \frac{1}{D^2} Tr \left\{ \delta K(W^c_n) R^c_{nlj} \right\} Tr \left\{ \frac{\partial}{\partial W} \delta K(W^c_n) R^c_{nlj} \right\}.$$

---

[6] We note that this problem doesn't occur if we use the kernel $K_c = K_D$ in this case the divergent part of (3.16) completely cancel (3.11).

Because (taking in account (3.8),(3.15) and note 2):

$$\left\langle \frac{\partial}{\partial W} \delta K \right\rangle_{nlj} \Bigg|_{one\ loop\ \log\left(\frac{\Lambda^2}{m^2}\right)\ terms} = \left\langle \frac{\partial}{\partial W} K_{s.e.}^{div} \right\rangle_{nlj} \qquad (3.17)$$

$$= \frac{\alpha}{\pi} \left(-\frac{1}{4} \log\left(\frac{\Lambda^2}{m^2}\right) \right) \quad \cdot \quad \left\langle -i\gamma^0 \right\rangle_{nlj}$$

$$= \frac{\alpha}{\pi} \left(-\frac{1}{4} \log\left(\frac{\Lambda^2}{m^2}\right) \right) \quad \cdot \quad \left(1 + \left\langle \frac{\partial}{\partial W} K_c \right\rangle_{nlj} \right)$$

$$= \frac{\alpha}{\pi} \left(-\frac{1}{4} \log\left(\frac{\Lambda^2}{m^2}\right) \right) \quad \cdot \quad \left(1 + O\left((Z\alpha)^2\right) \right) ;$$

one has:

$$\left\langle K_D - K_c \right\rangle_{nlj} \left\langle \frac{\partial}{\partial W} \delta K \right\rangle_{nlj} \Bigg|_{one\ loop\ \log\left(\frac{\Lambda^2}{m^2}\right)\ terms} =$$

$$\left(\langle K_D \rangle_{nlj} - \langle K_c \rangle_n \right) \frac{\alpha}{\pi} \left(-\frac{1}{4} \log\left(\frac{\Lambda^2}{m^2}\right) \right) + O\left(\frac{\alpha}{\pi} (Z\alpha)^6 \log\left(\frac{\Lambda^2}{m^2}\right) \right). \quad (3.18)$$

Summing now (3.11), (3.16) and (3.18) up to order $\alpha/\pi(Z\alpha)^4$ all $\log(\Lambda^2/m^2)$ dependent terms cancel.

The contribution of $\left\langle \hat{K}_{s.e.} \right\rangle_{nlj}$ has been evaluated on the levels $n = 1, 2$ *exactly*, as a first step we show only the leading order terms:

$$\left\langle \hat{K}_{s.e.} \right\rangle_{1s} = \frac{\alpha}{\pi} (Z\alpha)^2 \left[-\frac{3}{8} + 2\log(Z\alpha) + 2\log(2) \right], \qquad (3.19)$$

$$\left\langle \hat{K}_{s.e.} \right\rangle_{2s} = \frac{\alpha}{\pi} (Z\alpha)^2 \left[-\frac{1}{32} + \frac{1}{2}\log(Z\alpha) \right], \qquad (3.20)$$

$$\left\langle \hat{K}_{s.e.} \right\rangle_{2p} = \frac{\alpha}{\pi} (Z\alpha)^2 \left[-\frac{11}{96} + \frac{1}{2}\log(Z\alpha) \right]. \qquad (3.21)$$

Here and in the following the indication of just the nl levels means there isn't contribution to the splitting in the j levels.

The $\alpha/\pi(Z\alpha)^2$ and $\alpha/\pi(Z\alpha)^2 \log(Z\alpha)$ terms in (3.19)-(3.21) are expect to vanish. This occur because the Dirac levels (2.24) of order $(Z\alpha)^4$ are

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completely given by the graphs discussed in the previous section. This is a well known feature of the Feynman gauge (see e.g [18]); it decreases the speed of convergence of the perturbative series by generating spurious terms of low order that only at the end of *all* calculation (up to requested order in $Z\alpha$) must cancel each other.

Up to order $\alpha/\pi(Z\alpha)^2$ the contributions to the energy levels $n = 1, 2$ of $\left\langle \hat{K}_{ver} \right\rangle_{nlj}$ are:

$$\left\langle \hat{K}_{ver} \right\rangle_{1s} = \frac{\alpha}{\pi}(Z\alpha)^2 \left[\frac{39}{8} - 2\log(Z\alpha) - 10\log(2) \right], \qquad (3.22)$$

$$\left\langle \hat{K}_{ver} \right\rangle_{2s} = \frac{\alpha}{\pi}(Z\alpha)^2 \left[\frac{1003}{288} - \frac{1}{2}\log(Z\alpha) - \frac{16}{3}\log(2) \right], \qquad (3.23)$$

$$\left\langle \hat{K}_{ver} \right\rangle_{2p} = \frac{\alpha}{\pi}(Z\alpha)^2 \left[\frac{2009}{864} - \frac{1}{2}\log(Z\alpha) - \frac{32}{9}\log(2) \right]. \qquad (3.24)$$

These contribution only cancel the $\alpha/\pi(Z\alpha)^2 \log(Z\alpha)$ spurious terms of (3.19)-(3.21). To obtain the full cancellation of all $\alpha/\pi(Z\alpha)^2$ terms which are in (3.19)-(3.21) it is necessary to calculate the leading order of the remaining graphs of fig.4:

$$
\begin{aligned}
\langle K_{ladder} \rangle_{nlj} &\equiv \frac{1}{D} \sum_{j=1}^{\infty} \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2 + i\epsilon} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \\
&\quad \int \frac{d^3x_1}{(2\pi)^3} \int \frac{d^3x_2}{(2\pi)^3} ... \int \frac{d^3x_j}{(2\pi)^3} Tr\left\{ (-ie\gamma^\mu) \right. \\
&\quad G_0(W_n^c - k^0, \vec{q} - \vec{k}) K_D(\vec{x}_1, \vec{q}) G_0(W_n^c - k^0, \vec{x}_1 - \vec{k}) ... \\
&\quad G_0(W_n^c - k^0, \vec{x}_j - \vec{k}) K_D(\vec{x}_j, \vec{p}) G_0(W_n^c - k^0, \vec{p} - \vec{k}) \\
&\qquad\qquad\qquad\qquad\qquad \left. (-ie\gamma^\nu) R_{nlj}^c(\vec{p}, \vec{q}) \right\} \\
&\approx \frac{1}{2n} \frac{\alpha}{\pi}(Z\alpha)^2 \int_0^\infty dk \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} R_{nl}(p) R_{nl}(q) \\
&\quad P_l\left(\frac{\vec{p} \cdot \vec{q}}{pq} \right) \frac{1}{(\vec{p} - \vec{q})^2} \frac{p^2 + q^2 + 2}{(k + p^2 + 1)(k + q^2 + 1)} \qquad (3.25)
\end{aligned}
$$

From the explicit calculation of (3.25) on the levels $n = 1$ and $n = 2$ it follows:

$$\langle K_{ladder} \rangle_{1s} = \frac{\alpha}{\pi}(Z\alpha)^2 \left[-\frac{9}{2} + 8\log(2) \right], \qquad (3.26)$$

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$$\langle K_{ladder} \rangle_{2s} = \frac{\alpha}{\pi}(Z\alpha)^2 \left[-\frac{497}{144} + \frac{16}{3}\log(2) \right], \tag{3.27}$$

$$\langle K_{ladder} \rangle_{2p} = \frac{\alpha}{\pi}(Z\alpha)^2 \left[-\frac{955}{432} + \frac{32}{9}\log(2) \right]. \tag{3.28}$$

These contributions eliminate, as we have announced, the remaining $\alpha/\pi(Z\alpha)^2$ terms from (3.19), (3.20), (3.21) and (3.22), (3.23), (3.24).

At the end of this section we present the complete (but for short up to order $(\alpha/\pi)(Z\alpha)^7$) contribution of $\langle \hat{K}_{s.e.} \rangle_{nlj}$ without the terms cancelled by the leading contribution of $\langle \hat{K}_{ver} \rangle_{nlj}$ and $\langle K_{ladder} \rangle_{nlj}$:

$$\begin{aligned}
\Delta E_{1s} &\equiv \left\langle \hat{K}_{s.e.} + \left(\hat{K}_{ver} + K_{ladder} \right)\big|_{leading} \right\rangle_{1s} = \\
&\quad \frac{\alpha}{\pi}(Z\alpha)^4 \left[\frac{37}{16} + 7\log(Z\alpha) + 7\log(2) \right] \\
&+ \frac{\alpha}{\pi}(Z\alpha)^5 \left[\frac{20}{9\pi} - 6\pi - \frac{1}{\pi}I - \frac{14}{3\pi}\log(2) \right] \\
&+ \frac{\alpha}{\pi}(Z\alpha)^6 \left[\frac{983}{64} - \frac{45}{4}\log(Z\alpha) - \frac{45}{4}\log(2) \right] \\
&+ \frac{\alpha}{\pi}(Z\alpha)^7 \left[\frac{1237}{225\pi} - \frac{5}{4\pi}I - \frac{463}{30\pi}\log(2) \right];
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
\Delta E_{2s} &\equiv \left\langle \hat{K}_{s.e.} + \left(\hat{K}_{ver} + K_{ladder} \right)\big|_{leading} \right\rangle_{2s} = \\
&\quad \frac{\alpha}{\pi}(Z\alpha)^4 \left[\frac{191}{256} + \frac{13}{16}\log(Z\alpha) \right] \\
&+ \frac{\alpha}{\pi}(Z\alpha)^5 \left[\frac{5}{18\pi} - \frac{3\pi}{4} - \frac{1}{8\pi}I - \frac{7}{12\pi}\log(2) \right] \\
&+ \frac{\alpha}{\pi}(Z\alpha)^6 \left[\frac{2615}{12288} - \frac{509}{256}\log(Z\alpha) \right] \\
&+ \frac{\alpha}{\pi}(Z\alpha)^7 \left[\frac{133}{7200\pi} + \frac{3\pi}{4} - \frac{53}{128\pi}I - \frac{607}{960\pi}\log(2) \right];
\end{aligned} \tag{3.30}$$

$$\Delta E_{2p} \equiv \left\langle \hat{K}_{s.e.} + \left(\hat{K}_{ver} + K_{ladder} \right)\big|_{leading} \right\rangle_{2p} =$$

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$$\frac{\alpha}{\pi}(Z\alpha)^4 \left[-\frac{43}{768} + \frac{5}{16}\log(Z\alpha) \right]$$
$$+ \quad \frac{\alpha}{\pi}(Z\alpha)^6 \left[\frac{9181}{36864} - \frac{329}{768}\log(Z\alpha) \right]$$
$$+ \quad \frac{\alpha}{\pi}(Z\alpha)^7 \left[\frac{23}{450\pi} - \frac{\pi}{4} + \frac{1}{8\pi}I + \frac{1}{20\pi}\log(2) \right];$$

$$\tag{3.31}$$

where we have defined [7]:

$$I = \frac{\text{Li}_2(3 - 2\sqrt{2}) + 2\text{Li}_2(1) + \log^2(1 + \sqrt{2})}{\sqrt{2}}. \tag{3.32}$$

4 Conclusions

We have shown how it is possible to perform a perturbative expansion of the hydrogenic bound state two-points Green function by choosing an appropriate zeroth-order kernel. We have written the corresponding zeroth-order solution and the perturbative expansion for the bound state energy levels. Our approach is very transparent and unambiguous in the sense that in this

---

[7] The presence of the dilogarithms in (3.32) is not surprising, being a standard feature of this kind of calculations. For example, such terms can have their origin from integrals of the kind

$$f(W) = -\int_0^1 \frac{dy}{y} \log(y^2 + 2Wy + 1),$$

obtained after the change of variables $p + E_p = y$ and after some rationalizations. It is easy to see how

$$\frac{df(W)}{dW} = -\frac{1}{\sqrt{1 - W^2}}\text{atan}\frac{\sqrt{1 - W^2}}{W},$$

$$f(1) = 2\text{Li}_2(-1) = -\frac{\pi^2}{6},$$

so that

$$f(W) = -\frac{\pi^2}{6} + \frac{1}{2}\text{atan}^2\frac{\sqrt{1 - W^2}}{W}.$$

Other terms come from similar integrals giving at the end, after an expansion in $Z\alpha$, the results (3.19)-(3.21).

way one knows exactly what he is discarding and what he is keeping, and each approximation is just referred to neglecting some Feynman graph. Then we have expanded perturbatively the self-energy, and we have calculated the exact contribution of the first graph in the expansion, here presented up to order $(\alpha/\pi)(Z\alpha)^7$. The way is still long and hard: the next terms, which contribute to the self-energy are the one-Coulomb exchange diagram and the sum (from two up to infinity) of *all* many-Coulomb exchanges. While for the one-Coulomb exchange one may think to proceed in the calculation by brute force, for the sum of the many-Coulomb exchange graphs there are the difficulties to treat the Schwinger integral which compare in the Dirac-Coulomb propagator's expansion. These difficulties are essentially the same which are present in the other bound-state problems in QED. Therefore, the solution of these difficulties is not only important in order to have progress in the hydrogenic atoms energy levels calculation but it will make possible, in particular, an *ex novo* calculation in our formalism of the positronium energy levels. Along this direction we are going proceed further on.

Acknowledgments

A   Appendix

All graphs of fig.3 give contributions to the energy level of the same order in $Z\alpha$, so that such an expansion of the Dirac-Coulomb propagator g_D is wrong from the perturbative point of view. In section 2 we have discussed how to treat perturbatively in our formalism the Dirac-Coulomb propagator. In this appendix we review the alternative method proposed by J. Sapirstein [7] (see [19] for an application). This method is more artificial than ours which follows as a natural consequence from the formalism. Therefore one expects to have some additional analytical problems to implement the following expansion for g_D into the energy expansion (2.7).
The idea is to isolate from g_D a scalar part s_D and then to write a perturba-

tive expansion for s_D. We define s_D as:

$$
\begin{aligned}
g_D(W;\vec{p},\vec{q}) &\equiv (\gamma^0 W - \vec{p}\cdot\vec{\gamma} + m)s_D(W;\vec{p},\vec{q}) \\
&+ \frac{Z\alpha\gamma^0}{2\pi^2}\int d^3k \frac{1}{(\vec{p}-\vec{k})^2}s_D(W;\vec{k},\vec{q}) \\
&= (iG_0^{-1} - \tilde{V}_c\gamma^0)s_D.
\end{aligned}
\tag{A.1}
$$

This means also that $s_D = -ig_D^2$.

In the following, in order to make the notation short, we will write all formulas as in (A.1), i.e. without the explicit indication of the momenta and their integration. Substituting (A.1) into the Dirac-Coulomb equation (2.21) we obtain the equation for s_D:

$$
\begin{aligned}
s_D &= g_0(1 + k\ s_D), \tag{A.2}\\
k &= k_c + \delta k, \tag{A.3}
\end{aligned}
$$

where:

$$
\begin{aligned}
g_0(W;\vec{p},\vec{q}) &\equiv -\frac{i}{E_p^2 - W^2}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}), \\
k_c(W;\vec{p},\vec{q}) &\equiv 8\pi i\ Z\alpha\frac{W}{(\vec{p}-\vec{q})^2}, \\
\delta k(\vec{p},\vec{q}) &\equiv 2\pi^2 i(Z\alpha)^2\frac{1}{|\vec{p}-\vec{q}|} + 4\pi i(Z\alpha)\frac{\gamma^0\ \vec{\gamma}\cdot(\vec{p}-\vec{q})}{(\vec{p}-\vec{q})^2} \\
&\equiv \delta k_1 + \delta k_2.
\end{aligned}
$$

We can consider (A.2) and (A.3) as the analogous of (2.2), (2.3) respectively. It is important to remark that in (A.3) δk is really a "perturbation" of k_c in the sense that all the contributions which arise from δk are of higher order in $Z\alpha$ than the contributions which arise from k_c. Finally we can write as in (2.6) the Bethe-Salpeter equation for s_D:

$$
s_D = g_c(1 + \delta k\ s_D), \tag{A.4}
$$

where g_c is given by (see (2.13)):

$$
g_c(W;\vec{p},\vec{q}) = g_0(W;\vec{p},\vec{q}) - \frac{1}{E_p^2 - W^2}k_c(W;\vec{p},\vec{q})\frac{1}{E_q^2 - W^2}
$$

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$$\cdot \quad \left[1 + Z\alpha \frac{W}{\sqrt{m^2 - W^2}} \int_0^1 d\rho \frac{\rho^{\frac{-WZ\alpha}{\sqrt{m^2 - W^2}}}}{\rho + (1-\rho)^2 \frac{(E_p^2 - W^2)(Eq^2 - W^2)}{4(m^2 - W^2)(\vec{p} - \vec{q})^2}} \right]$$

$$\equiv \quad g_0 + g_1^c + g_M^c. \tag{A.5}$$

From (A.5), (A.4) and (A.1) we obtain a perturbative expansion for the Dirac-Coulomb propagator g_D:

$$g_D = (iG_0^{-1} - \tilde{V}_c \gamma^0) \Big[g_c + g_c \; \delta k \; g_c +$$

$$+ \quad g_c \; \delta k \; g_c \; \delta k \; (g_c + g_c \; \delta k \; g_c + ...) \Big]. \tag{A.6}$$

If we want to isolate from this expansion the first two terms of fig.3, we observe that:

$$iG_0^{-1} g_0 = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) G_0, \tag{A.7}$$

$$iG_0^{-1} g_1^c + iG_0^{-1} g_0 \; \delta k_1 \; g_0 - \tilde{V}_c \gamma^0 g_0 = G_0(-i\gamma^0 \tilde{V}_c) G_0, \tag{A.8}$$

and hence we rewrite (A.6) as:

$$g_D = G_0 + G_0(-i\gamma^0 \tilde{V}_c) G_0 + iG_0^{-1} g_M^c$$
$$+ \quad iG_0^{-1} g_0 \; \delta k_2 \; g_0 - \tilde{V}_c \gamma^0 (g_1^c + g_M^c) - \tilde{V}_c \gamma^0 g_0 \; \delta k \; g_0$$
$$+ \quad (iG_0^{-1} - \tilde{V}_c \gamma^0) \Big[(g_1^c + g_M^c) \; \delta k \; g_c + g_c \; \delta k \; (g_1^c + g_M^c)$$
$$+ \quad g_c \; \delta k \; g_c \; \delta k \; (g_c + g_c \; \delta k \; g_c + ...) \Big]. \tag{A.9}$$

We observe that the leading order of the contribution of the remaining graphs of fig.3 arises from $iG_0^{-1} g_M^c$.

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